

Generic structure of externally driven tearing modes instabilities

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Abstract. A major limit to steady state and advanced high β_p operation of tokamaks of reactor class is due to the onset of *tearing modes* that develop magnetic and may cause loss of energy confinement or a major disruption. Here the structure of a classical problem about the effects of external control helical fields is analysed and it is shown to offer a general paradigm of response of low order classical and neoclassical tearing modes to a wide class of external perturbations. New results of principle on the structural stability of the response model are obtained, leading to a clear interpretation of the role of “seed islands” in the onset of neo-classical tearing modes and the role of finite ion larmor radius corrections to Ohm’s law.

PACS. 52.35.Py Macroinstabilities (hydromagnetic, e.g., kink, fire-hose, mirror, ballooning, tearing, trapped-particle, flute, Rayleigh-Taylor, etc.) – 52.25.Fi Transport properties – 52.35.Mw Nonlinear phenomena: waves, wave propagation, and other interactions (including parametric effects, mode coupling, ponderomotive effects, etc.)

1 Introduction

The prospects of a realistic thermonuclear reactor based on tokamaks rest as much on the reaching suitable plasma energy and confinement target values, as on the reliability of operation of a tokamak for an *economically significant* duration of the discharge. The attainment of thermonuclear regimes, with high temperature, very long resistive diffusion time scales, high β_p effects, is finally dependent on a major problem of tokamak physics and operation, namely the control of the inevitable instability of slow-growing resistive modes. Some basic questions raised in the early times of investigation of tokamak plasma stability [1–4] remain open and are worth reconsidering because they offer important paradigms of response embracing eventually issues relevant to advanced and neo-classical (high β_p) regimes. Referring to some very old and some recent experimental results, we reconsider here generic structural properties of the nonlinear *response* of single helicity low order tearing modes to the boundary conditions imposed by external active helical conductors with the pitch resonant with the closed field lines of magnetic surfaces having a rational safety factor q .

2 Structure of tearing modes models in tokamaks

Our interest is focussed on the structure of well-known Δ' analytical models of tearing modes stability [5]. Although

one should be aware of the role of mode rotation in the interaction with phased resonant external fields it is meaningful to consider a conceptual “stabilisation” scenario in which the phase difference is kept constant by some feedback loop and the sole control parameter is the amplitude of the external field. Where appropriate we support the analytical arguments with results of numerical calculations from our non-linear RMHD code which solves the poloidal flux (ψ) and stream function (u) time evolution equations (Eqs. (1, 2) with external forced boundary conditions. For our purposes it is adequate to assume that the plasma is of uniform density, of circular cross-section of radius a , surrounded by a thin coaxial resistive wall and an initial equilibrium current profile $J_0(r)$ specified in terms of the safety factor $q(r)$. At radius b , $a < b \ll d$, in vacuum, between the plasma minor radius a and the vessel radius d , the active coils system is modelled by an ideal helical surface current $I_E = I_0 \text{Re}(e^{i(m\vartheta - \phi_E)})$ with pitch ratio m/n resonant with the safety factor q at $r = r_s$. Non-linear effects are considered just for the modification of the equilibrium poloidal flux, ensuring mode saturation (second order terms such as $\mathbf{v} \cdot \text{grad} \mathbf{v}$ are therefore neglected)

$$\frac{\partial \psi}{\partial t} + \mathbf{B} \cdot \nabla u = -\eta_0 j_z + E_0 \quad (1)$$

$$\rho_0 \frac{\partial \nabla^2 u}{\partial t} = \mathbf{B} \cdot \nabla j_z + \nu \nabla^4 u. \quad (2)$$

The magnetic field and plasma velocity are written in terms of the magnetic flux function ($\psi \equiv \psi_0 + \tilde{\psi}$) and

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stream function u as $\mathbf{B} = B_{z0}\hat{\mathbf{z}} + \nabla \times (\psi\hat{\mathbf{z}})$ and $\mathbf{v} = \nabla \times (u\hat{\mathbf{z}})$. The subscript 0 denotes equilibrium quantities and ρ_0 , ν , η_0 and E_0 are respectively, the plasma density, viscosity, resistivity and the equilibrium toroidal electric field. The numeric simulations were done with a normalised version of equations (1, 2) where the following normalisation was used: $\psi \rightarrow \psi/aB_{z0}$, $u \rightarrow u(\tau_A/a^2)$, $t \rightarrow t/\tau_A$, $x \rightarrow r/a$ where $\tau_A = a/\sqrt{B_{z0}^2/\mu_0\rho_0}$ is the Alfvén time, $\tau_\nu = \rho_0 a^2/\nu$ is an effective viscous time, $\tau_R = \mu_0 a^2/\eta_0$ is the resistive diffusion time and $S = \tau_R/\tau_A$. A magnetic Reynolds number S of the order 10^4 and a very low viscosity are used, since they don't affect the overall results.

The boundary conditions matching the external driving fields are enforce continuity of the normal component of the magnetic field, and of the tangential component of $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ and vanishing of the normal component of the velocity at the plasma vacuum interface, as appropriate for a resistive plasma with null current density at the edge [6].

We first consider an initial equilibrium q profile that is linearly unstable to the tearing instability (*i.e.* with a positive jump of the logarithmic derivative Δ' of the flux perturbation at the rational surface) for mode numbers ($m = 2$, $n = 1$), in presence of an applied helically resonant current. This case, was analysed first by Monticello *et al.* [7] and led to the theoretical identification of the so-called “flip” instability, in which the application of a constant boundary condition (a helical current, referred to as $I_{E,\text{thresh.}}$ with a negative value) meant to restore $\Delta' = 0$, led on the contrary to another unstable state with $\Delta' > 0$. This is a “flip” state because the value of the reconnected flux ψ_s at the rational surface, consistent with the boundary conditions, changes sign, and this can be interpreted as a shift of π of the equilibrium position of the “O” point of the topology tearing perturbation. In a conventional description of such events, the numerical solution of the standard linear tearing mode equation with different negative current boundary conditions shows, that the equation $\Delta'(w) = 0$ (where $w = \sqrt{\psi_s/k}$, and $k = (Rq'/16q^2)_{r_s}$) can have one, two or no real roots. For $I_E > I_{E,\text{thres.}}$ the larger root is the stable saturated island width; for $I_E < I_{E,\text{thres.}}$ no roots exist and $\Delta'(w) < 0$ therefore in this description the island is bound to decrease indefinitely, while for $I_E \equiv I_{E,\text{thres.}}$ the equilibrium state is unstable and the island should also be *prone* to decrease. A first outcome of the non-linear time dependent calculations of the reconnection process is that the mode “flips” if the time trace of ψ_s has an inflection point $d^2\psi_s/dt^2 = 0$ with $d\psi_s/dt < 0$ and *not* exactly when the linear Δ' is annihilated (this small difference attenuates for viscous plasmas). Incidentally, in experiments it should be far easier to detect the flip condition from $d^2\psi_s/dt^2 = 0$ than $\Delta' = 0$.

The most important message of the non-linear numerical code is given by the global response curve of the system to the external control parameter $|I_E|$, shown in Figure 1 presenting the dependence of the modulus of the *state variable* ψ_s on the *control variable* $|I_E|$. As $|I_E|$ varies the saturated reconnected flux can be squeezed toward a

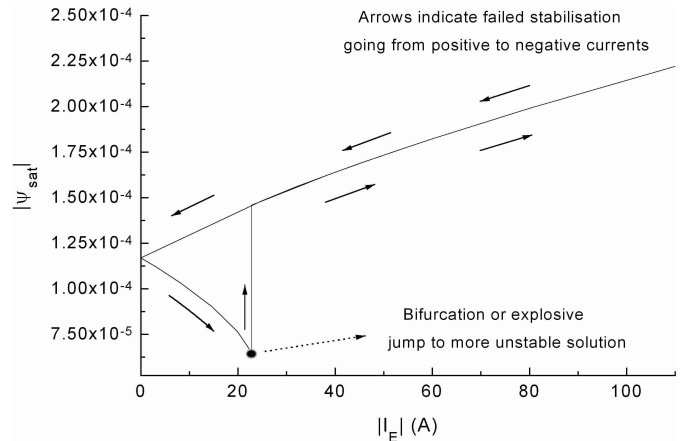


Fig. 1. Absolute value of reconnected flux $|\psi_s|$ vs. absolute value of external control current, $|I_E|$ calculated by a full non-linear tearing mode code in large R/a approximation.

minimum, at a critical value of $|I_E|$ where it pops back to an *amplified* saturated value. This result is the basis for a comparison and new interpretation of the general analytical form of forced (non-linear) flux reconnection at a rational surface $r = r_s$, in the magnetic island instantaneous frame:

$$\frac{\tau_R}{r_s} \frac{d\psi_s}{dt} = 2\sqrt{k|\psi_s|} \left\{ \Delta'_0(\psi_s) + \frac{\lambda}{\psi_s} \right\} \equiv \frac{\tau_R}{r_s} f(\psi_s, \lambda) \quad (3)$$

where the control parameter λ is proportional to the external current. Equation (3) can be identified as a modified Rutherford equation. Throughout the paper, the magnetic field shear parameter (k) is not considered as a control parameter but just a constant relating the reconnected flux at the q rational surface to the corresponding magnetic island width. For a theoretical interpretation one can consider first the simplest case of a generic non-linear modification of the equilibrium current profile [8]

$$\Delta'_0(\psi_s) \equiv \Delta'_0 \left(1 - w_{\text{sat}}^{-1} \sqrt{|\psi_s|/k} \right) + \dots \text{ with } \Delta'_0 > 0. \quad (4)$$

This describes satisfactorily the spontaneous reconnection process up to saturation in an ohmic tokamak plasma. It is important, and not trivial, to note that in response to an external control of amplitude λ the value of the reconnected flux $\psi_s(t)$ can be either positive or negative at any given time. It is also important and not fortuitous that the control term, that physically represents an externally driven e.m.f. with the purpose of counteracting the spontaneous flux change, has a dependence $\propto \psi_s^{-1}$. This encompasses both the cases of modification of the boundary conditions by currents localised out of the plasma and the case of currents driven, say by radio frequency (rf), within a magnetic island. The mode evolution described by the equation

$$\frac{d\psi_s}{dt} = f(\psi_s(t), \lambda(t))$$

is a non-linear, time dependent problem with an arbitrarily time dependent inhomogeneous term. A first essential

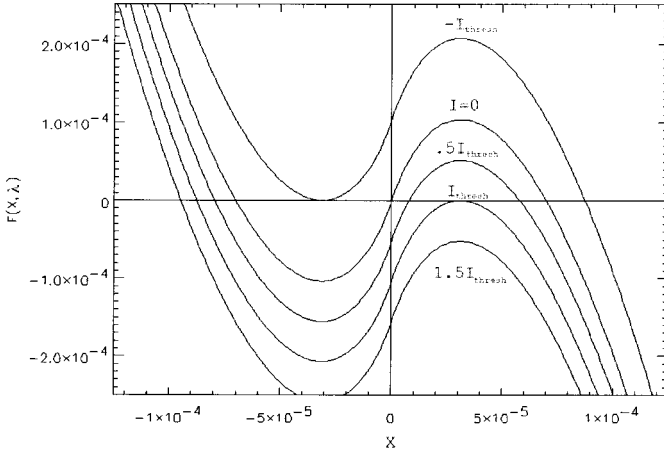


Fig. 2. Rate function $F(X = \psi_s, \lambda)$ vs. the state variable X for “ohmic” tearing mode model (7) at different values of the control variable λ . Intersections with the X -axis represent the equilibrium states (stable and unstable) of the system.

analysis can be however carried out for constant values of λ , inspecting the complex roots of $F = \sqrt{|\psi_s|}f(\psi_s, \lambda)$ that has the same sign of $d\psi_s/dt$. For the model (4) Figure 2 shows the behaviour, as λ is varied, of the three roots that represent the equilibrium values of reconnected flux, corresponding to finite size magnetic islands. The zeros of the function $F(-\lambda)$ are the opposite of those of $F(\lambda)$. In the traditional discussion [9] of the Rutherford equation (for $\lambda > 0$) only the two real positive roots corresponding to an *unstable* island width and a saturated *stable* one are visible. Presently we draw attention also on the third, real negative root, corresponding also to *another* stable island value. At the critical values (ψ_{sc}, λ_c) where $F(\psi_{sc}, \lambda_c) = 0$, $F_{\psi}(\psi_{sc}, \lambda_c) = 0$ the two (positive) roots coalesce and explosive transition to the third root occurs, that corresponds to an island value larger than the saturated value. This describes the mechanism of the so-called “flip” instability *rigorously*, with a consistent formalism that allows treatment of problems with similar structure. In detail for the benchmark model (4) the fixed point is $\psi_{sc} = (4/9)w_s^2$, $\lambda_c = (4/27)\Delta'_0 w_s^2$ and the lowest order expansion suitable to study the local non-linear stability near the fixed point is

$$F(\psi, \lambda) \cong (\lambda - \lambda_c)F_{\lambda}(\psi_{sc}, \lambda_c) + \frac{1}{2}(\psi - \psi_{sc})^2 F_{\psi\psi}(\psi_{sc}, \lambda_c) + \dots \quad (5)$$

This expression with variables re-scaling and up to a diffeomorphic transformation, leads to the identification of a local generic fold equilibrium manifold of normal form $g(z, \mu) = z^2 - \mu$ in the plane ($z \propto (\psi - \psi_{sc})$, $\mu \propto (\lambda - \lambda_c)$). This form leads locally to a tangent bifurcation [10–12] with explosive rate of departure

$$\frac{d(\psi_s - \psi_{sc})}{dt} = \mp(\psi_s - \psi_{sc})^2$$

from the singular points to the “flipped state”. The overall response of the system to the control parameter (cur-

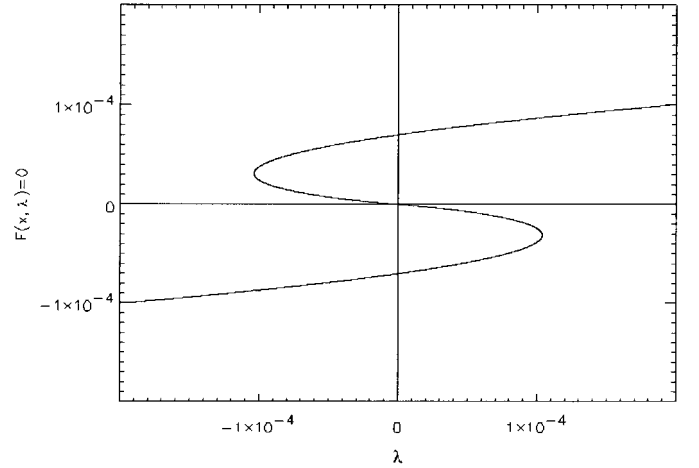


Fig. 3. Equilibrium manifold $F(X = \psi_s, \lambda) = 0$ vs. the control variable λ for “ohmic” tearing mode model (7). The tangent bifurcation points are apparent.

rent) is summarised in Figure 3 where the amplitude of the flux ψ_s is plotted against the (positive and negative) values of controlling current. The S shape of the locus of the roots of $F(\psi_{sc}, \lambda_c) = 0$ indicates tangent bifurcations at positive and negative values of λ . By transforming this curve (by reflection of the left-hand and bottom half-planes) in a plot of $|\psi_s|$ versus $|\lambda|$ one obtains exactly the response curve Figure 1 of the non-linear code. An important question from singularity theory concerns the structural stability of the scalar bifurcation problem $F(\psi_{sc}, \lambda_c) = 0$ as $F_{\psi}(\psi_{sc}, \lambda_c) = 0$ parameters are varied and it is well known that the fold bifurcation is structurally stable [10–12].

In present day tokamak performance at relatively large values of β_p a significant problem is the onset of neoclassical tearing modes (NTM) due to the reduction of the pressure gradient driven bootstrap current over a “seed” island w_s formed at low q rational surfaces. This motivates also the investigation of static or quasi-static “error” field compensation systems that can reduce the danger of formation of seed islands by error field driven reconnections. In these specific circumstances, for finite plasmas with a non negligible ratio β_p of kinetic pressure over poloidal magnetic energy density, the form of the instability parameter $\Delta'_0(\psi_s)$ can be written singling out in a basic form the destabilising neoclassical [13] and of the ion polarisation current contributions [14,15], resulting from the requirement $\mathbf{B} \cdot \nabla(J_{\parallel}/B) = -\nabla_{\perp} \cdot \mathbf{J}_{\perp}$ up to order $(\rho_{i\theta}/W)^2$

$$\Delta'_0(\psi_s) \equiv -|\Delta'_0| + \beta_{\theta} a_b \frac{\sqrt{|\psi_s|/k}}{|\psi_s|/k + w_d^2} - \beta_{\theta} a_p \frac{k^{3/2}}{|\psi_s| \sqrt{|\psi_s|}} \quad (6)$$

where $a_b = O(1)$ is a net, effective, profile dependent coefficient that includes the bootstrap current contribution, the stabilising Greene Glasser and Johnson term and the magnetic well term due to cross-section triangular deformations [18,19] and $a_p \ll 1$ is the ion polarisation current effect [14,15], w_d is the diffusive cut-off island width described in reference [9] “ χ -model”. These expressions are

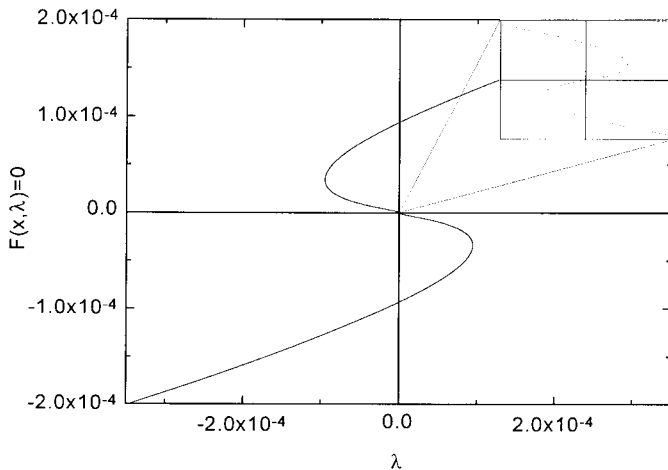


Fig. 4. Equilibrium manifold $G(X = \psi_s, \lambda) = 0$ vs. the control variable λ for “neoclassical” tearing mode model (6). The tangent bifurcation points and the local *generic structure* are apparent in the insert.

most often considered in the context of extensions of the Rutherford equations written in terms of the *strictly positive* island width $w \propto \sqrt{|\psi_s|}$.

The bifurcation analysis is extended to the neoclassical case [13] (see Eq. (7)) considering the zeros of

$$\begin{aligned}
 G &= \psi_s^2 \sqrt{|\psi_s|} \frac{d\psi_s}{dt} \\
 &\equiv \psi_s^3 \left[-|\Delta'_0| + \beta_\theta a_b \frac{\sqrt{|\psi_s|/k}}{|\psi_s|/k + w_d^2} \right. \\
 &\quad \left. - \beta_\theta a_p \frac{k^{3/2}}{|\psi_s| \sqrt{|\psi_s|}} \right] + \lambda \psi_s^2
 \end{aligned} \quad (7)$$

that has the same sign of $d\psi_s/dt$ but also the root $\psi_s = 0$ that is *not* an equilibrium state of the original equation. Applying the same procedure to the case with vanishing ion polarisation current ($a_p = 0$), we obtain the S shaped locus of the roots of G (Fig. 4) vs. λ , similar to Figure 3 and therefore generic and structurally stable. Inspection of Figure 4 shows also a known mechanism of trigger of a neoclassical mode by external (error) helical fields or equivalent source [9]. In a rotating plasma, starting from the *non equilibrium* state $\psi_s = 0$ (no island) as soon as the control parameter (external current) changes, the mode amplitude increases passing through a sequence of very small saturated states. As shown in reference [20] the forced reconnection process evolves in such a way that the phase difference $\Delta\varphi$ between the driving field and the mode tends to a steady value in the range $0 \ll \Delta\varphi < \pi/2$. This just reduces the effective value of the control parameter $\lambda \equiv \lambda(\Delta\varphi)$. However when a threshold in the $\lambda(\Delta\varphi)$ is passed the system undergoes a tangent bifurcation and the mode explodes toward a high saturated equilibrium amplitude. It has already been shown in reference [9] that the bifurcation for the NTM case can be triggered by error field amplitudes much lower than that required to amplify Δ' stable modes without neoclassical

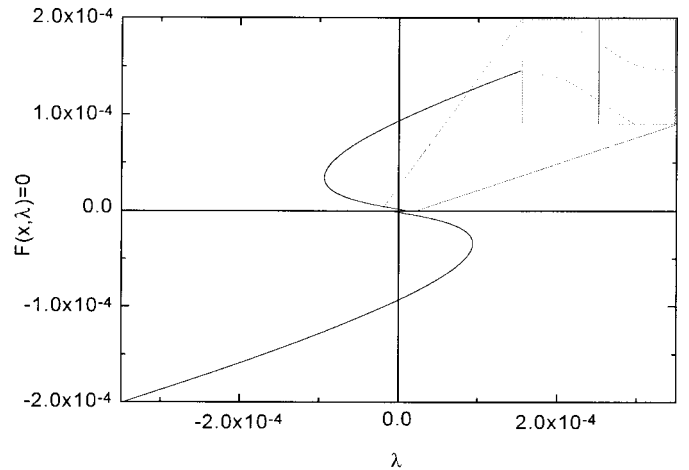


Fig. 5. Equilibrium manifold $G(X = \psi_s, \lambda) = 0$ vs. the control variable λ for “neoclassical model with ion polarisation current”. The insert shows the *non generic structure* due to $a_p \neq 0$.

effects. Inclusion of rotation increases the threshold for bifurcation substantially, by a factor $1/|\cos(\Delta\varphi)|$. When the model (6) includes a finite, albeit small, ion polarisation current term $a_p \neq 0$, of either sign, the situation is substantially changed. The locus of the roots of $G(\lambda) = 0$ shown in Figure 5 is *not generic* and structurally stable [12] as in the previous cases. The flip bifurcation is prevented except at vanishing of the ion polarisation term $a_p \propto (r_s/R)^{3/2} \rho_{i\theta}^2 \omega_{*pe}^2 (\omega - \omega_E)(\omega - \omega_T)$, where ω_E is the electric drift frequency and ω_T is the natural mode frequency defined in references [14, 15].

The role of this term, that takes into account the low collisionality reduction of the neoclassical effects, is widely debated [14–22]. The ion polarisation term persists even when the mode phase velocity vanishes under the effect of an error field and it prevents the bifurcation of NTM to be triggered by small error fields. Indeed, as long as $a_p \neq 0$, the small saturated states that proceed to a bifurcation are no longer accessible. To reach a saturated equilibrium, a threshold value, scaling as $\psi_{\text{thr}} \propto \lambda^{-1}$, must be overcome by an initial *seed island* [15, 16], induced by effects *other* than external error fields, linked possibly to coupling with other modes like the $(m = 1, n = 1)$, capable of producing a seed island larger than that corresponding to the first root shown in Figure 5. However in case of synchronism of the phase velocity of the external perturbation with the plasma natural rotation at the rational surface the *non-generic* state-control diagram of Figure 5 is destroyed and could allow in principle a *thresholdless onset* of NTM modes and incidentally also the flip instability. We remark however that the flip is unimportant in the case of control by an *un-phased* rf driven current localised within a strip encompassing the island, as a shift of π of the “O” point would not matter for the task of stabilisation. This is probably the situation met in some successful experiments [23, 24].

3 Conclusion

In conclusion we have given a simple but sufficiently rigorous picture of phenomena often described intuitively identifying the generic structure of tangent bifurcations in both classical tearing mode flipping and NTM triggering akin to mechanical beam buckling instabilities [10] well confirmed by non-linear RMHD calculations. The role of ion-polarisation current has been discussed from the point of view of structural stability irrespective of its sign. It is argued that NTM's are more likely destabilised by internal seed island formation than by external error fields. These questions of principle have immediate bearing on interpretation of experiments.

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